Finite-Volume Analysis for the Cahn-Hilliard equation with Dynamic boundary conditions

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Abstract This work is devoted to the convergence analysis of a finite-volume approximation of the 2D Cahn-Hilliard equation with dynamic boundary conditions. The method that we propose couples a 2d-finite-volume method in a bounded, smooth domain $\Omega \subset \mathbb{R}^2$ and a 1d-finite-volume method on $\partial\Omega$. We prove convergence of the sequence of approximate solutions. One of the main ingredient is a suitable space translation estimate that gives a limit in $L^\infty\left(0,T,H^1(\Omega)\right)$ whose trace is in $L^\infty\left(0,T,H^1(\partial\Omega)\right)$.

1 Introduction

We consider a smooth, connected and bounded domain $\Omega \subset \mathbb{R}^2$ and $\Gamma = \partial\Omega$ its boundary. Let $T > 0$ be given.
We are interested here in the following phase separation model in material science (refered to as the Cahn-Hilliard equation with dynamic boundary conditions):
Find the concentration of one of the two phases $c : (0,T) \times \Omega \to \mathbb{R}$ satisfying:

$$\begin{cases} \partial_t c = \Delta\mu, & \text{in } (0,T)\times\Omega; \\ \mu = -\Delta c + f_b'(c), & \text{in } (0,T)\times\Omega; \\ \partial_t c_{|\Gamma} = \Delta_\parallel c_{|\Gamma} - f_s'(c_{|\Gamma}) - \partial_n c, & \text{on } (0,T)\times\Gamma; \\ \partial_n \mu = 0, & \text{on } (0,T)\times\Gamma; \\ c(0,.) = c_0, & \text{in } \Omega; \end{cases} \tag{1}$$

where we have introduced an intermediate unknown: the chemical potential μ.

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The trace of c on Γ is noted $c_{|\Gamma}$, $\Delta_{\|}$ is the Laplace-Beltrami operator on Γ and ∂_n is the normal derivative at the boundary. The Cahn-Hilliard potentials f_b and f_s are nonlinear and they correspond respectively to the bulk and the surface free energy densities. In fact, several physical parameters should appear in the Cahn-Hilliard equation to account for physical properties of the studied system. However, these constants affect the readability of the problem. Thus, we have chosen to write the Problem (1) without these parameters.
We impose the homogeneous Neumann boundary condition for the chemical potential since no mass exchange can occur through the boundary. For many years, different authors studied the Cahn-Hilliard equation associated with the Neumann boundary condition for the order parameter c. In some cases, however, this condition is too restrictive to account for the interaction of the mixture with the walls. For this reason, physicists [4, 7] have recently introduced the Cahn-Hilliard system with dynamic boundary conditions (1). The associated free energy is the sum of a bulk free energy $\mathscr{F}_b$ and a surface free energy $\mathscr{F}_s$:

$$\mathscr{F}(c) = \underbrace{\int_\Omega \left(\frac{1}{2}|\nabla c|^2 + f_b(c)\right)}_{:=\mathscr{F}_b(c)} + \underbrace{\int_\Gamma \left(\frac{1}{2}|\nabla_{\|} c_{|\Gamma}|^2 + f_s(c_{|\Gamma})\right)}_{:=\mathscr{F}_s(c)}. \tag{2}$$

The dynamic boundary condition on c is chosen in such a way that the total free energy decreases with respect to time:

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathscr{F}(c(t,.)) = -\int_\Omega |\nabla \mu(t,.)|^2 - \int_\Gamma |\partial_t c_{|\Gamma}(t,.)|^2, \quad t \in [0,T[.$$

The potentials are supposed to satisfy standard assumptions:
Assumptions 1. :

- Dissipativity: $\liminf_{|c|\to\infty} f_b''(c) > 0$ and $\liminf_{|c|\to\infty} f_s''(c) > 0$.
- Polynomial growth for f_b: there exist $C_b > 0$ and a real $p \geq 2$ such that:

$$\left|f_b^{(m)}(c)\right| \leq C_b\left(1 + |c|^{p-m}\right), \quad m \in \{0,1,2\}.$$

A typical choice for f_b is the double-well function $f_b(c) = c^2(1-c)^2$.

From a theoretical point of view, this system has already been studied (see for exemple [6] and the references therein). From a numerical point of view, we have several results. In [4, 7], authors propose a finite-difference framework but without proof of convergence. A convergence result is proved in [2] with a finite element space semi-discretization, but in a slab with periodic boundary conditions in lateral directions. In this paper, we propose a convergence analysis of a finite-volume scheme for the space discretization. This method is well adapted to the coupling between the dynamics in the domain and those on the boundary by the flux term $\partial_n c$. Moreover, this kind of scheme preserves the mass and accounts naturally for the non-flat geometry of the boundary and the associated Laplace-Beltrami operator.

2 The discrete setting

2.1 The Finite-Volume meshes and notation

We recall that the domain Ω is not polygonal and that we have to solve an equation on Γ. Thus, our notation will be slightly different than the usual finite-volume notation (see for example [3]). Let $\mathfrak{M}$ be a decomposition of Ω into polygonal subsets (called control volumes and noted $\mathscr{K} \in \mathfrak{M}$) except perhaps for those on the boundary which can have a curved edge. For each control volume $\mathscr{K} \in \mathfrak{M}$, we associate a point $x_{\mathscr{K}}$ which satisfies the orthogonality condition (see [3]). The main differences with the usual finite-volume notation are those on the boundary mesh $\partial\mathfrak{M}$. This mesh is constituted of the set of curved edges σ on the boundary Γ. With respect to the interior mesh, we keep the usual notation (Figure 1) except for control volumes $\mathscr{K} \in \mathfrak{M}$ with one edge σ, at least, belonging to the boundary. In this case, $\mathscr{K}$ is not polygonal (σ is curved), we note $\widetilde{\mathscr{K}}$ the polygon formed by the vertices of $\mathscr{K}$ and by $m_{\widetilde{\mathscr{K}}}$ its Lebesgue measure. Note that $\widetilde{\mathscr{K}}$ may not be included in Ω. We will use two different notations for an element of $\partial\mathfrak{M}$: we note $\mathbf{e}$ when we consider it as a control volume belonging to $\partial\mathfrak{M}$ and we note σ when we consider it as the edge of an interior control volume $\mathscr{K} \in \mathfrak{M}$.
Let $\mathbf{e} \in \partial\mathfrak{M}$ be a boundary control volume and $\widetilde{\mathbf{e}}$ the corresponding chord. Their length are respectively noted $m_{\mathbf{e}}$ and $m_{\widetilde{\mathbf{e}}}$. If $\mathscr{K} \in \mathfrak{M}$ is the control volume such that $\mathbf{e} \subset \partial\mathscr{K}$, we set $x_{\mathbf{e}}$ as the intersection between Γ and the straight line passing through $x_{\mathscr{K}}$ and orthogonal to $\widetilde{\mathbf{e}}$. Let $y_{\mathbf{e}}$ be the intersection between the line $(x_{\mathscr{K}} x_{\mathbf{e}})$ and the chord $\widetilde{\mathbf{e}}$. We define $d_{\mathscr{K},\mathbf{e}}$ as the distance between the centers $x_{\mathscr{K}}$ and $y_{\mathbf{e}}$. Let $\mathscr{V}$ be the set of the vertices included in Γ and $d_{\mathbf{e},\mathbf{v}}$ be the distance between the center $y_{\mathbf{e}}$ and the vertex $\mathbf{v} \in \mathscr{V}$. For a vertex $\mathbf{v} = \mathbf{e}|\mathbf{e}' \in \Gamma$ which separates the control volumes $\mathbf{e}$ and $\mathbf{e}'$, we note $d_{\mathbf{e},\mathbf{e}'}$ the sum of $d_{\mathbf{e},\mathbf{v}}$ and $d_{\mathbf{e}',\mathbf{v}}$.

We can notice that all these quantities are computed by just knowing the coordinates of the vertices of the mesh in Γ. Thus, we do not need to know the equation of the boundary Γ.

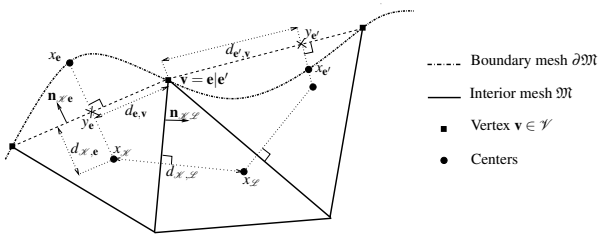


Fig. 1: Finite-volume meshes

We define the mesh size by: $h_{\mathscr{T}} = \sup\{\text{diam}(\mathscr{K}), \mathscr{K} \in \mathfrak{M}\}$. In the results below, all the constants depend on a certain measure of regularity of the mesh. This is classical and for the sake of simplicity, we do not give here its explicit value. In short, if this quantity is bounded when the mesh size tends to 0, this amounts to assume that the control volumes do not become flat when the mesh is refined.

2.2 Discrete unknowns

With respect to the time discretization, we introduce a positive integer N. Then, we uniformly partition the temporal interval $[0,T]$ with the time step: $\Delta t = T/N$. Thus, for $n \in \{0,\cdots,N\}$, we define $t^n = n\Delta t$.
For each time step t^n, we denote the concentration unknowns by $c_{\mathscr{T}}^n = (c_{\mathfrak{M}}^n, c_{\partial\mathfrak{M}}^n) \in \mathbb{R}^{\mathscr{T}}$ and the chemical potential unknowns by $\mu_{\mathscr{T}}^n = (\mu_{\mathfrak{M}}^n, \mu_{\partial\mathfrak{M}}^n) \in \mathbb{R}^{\mathscr{T}}$. Regarding the chemical potential, we have the homogeneous Neumann boundary condition; thus we can define the boundary unknown $\mu_{\partial\mathfrak{M}}^n \in \mathbb{R}^{\partial\mathfrak{M}}$ as follows:

$$\mu_{\mathbf{e}}^n = \mu_{\mathscr{K}}^n, \quad \forall \mathbf{e} \in \partial\mathfrak{M} \text{ such that } \mathbf{e} = \sigma \in \mathscr{E}_{\mathscr{K}}.$$

Finally, let $u_{\mathfrak{M}}^{\Delta t}$ (respectively $u_{\partial\mathfrak{M}}^{\Delta t}$) be the piecewise constant function in $]0,T[\times\Omega$ (respectively $]0,T[\times\Gamma$) such that for all $t \in [t^n, t^{n+1}[$:

$$u_{\mathfrak{M}}^{\Delta t}(t,x) = u_{\mathscr{K}}^{n+1} \text{ if } x \in \mathscr{K} \quad \text{and} \quad u_{\partial\mathfrak{M}}^{\Delta t}(t,x) = u_{\mathbf{e}}^{n+1} \text{ if } x \in \mathbf{e}.$$

2.3 Inner products and norms

- Discrete L^2 inner products: For all $u_{\mathfrak{M}}, v_{\mathfrak{M}} \in \mathbb{R}^{\mathfrak{M}}$ and $u_{\partial\mathfrak{M}}, v_{\partial\mathfrak{M}} \in \mathbb{R}^{\partial\mathfrak{M}}$, we define:

$$(u_{\mathfrak{M}}, v_{\mathfrak{M}})_{\mathfrak{M}} = \sum_{\mathscr{K}\in\mathfrak{M}} m_{\mathscr{K}} u_{\mathscr{K}} v_{\mathscr{K}} \quad \text{and} \quad (u_{\partial\mathfrak{M}}, v_{\partial\mathfrak{M}})_{\partial\mathfrak{M}} = \sum_{\mathbf{e}\in\partial\mathfrak{M}} m_{\widetilde{\mathbf{e}}} u_{\mathbf{e}} v_{\mathbf{e}}.$$

  The associated discrete L^2 norms are noted $\|u_{\mathfrak{M}}\|_{0,\mathfrak{M}}$ and $\|u_{\partial\mathfrak{M}}\|_{0,\partial\mathfrak{M}}$.
- Discrete H^1 semi-definite inner products: For all $u_{\mathscr{T}}, v_{\mathscr{T}} \in \mathbb{R}^{\mathscr{T}}$ and $u_{\partial\mathfrak{M}}, v_{\partial\mathfrak{M}} \in \mathbb{R}^{\partial\mathfrak{M}}$:

$$[\![u_{\mathscr{T}}, v_{\mathscr{T}}]\!]_{1,\mathscr{T}} = \sum_{\sigma\in\mathscr{E}_{int}} \frac{m_\sigma}{d_{\mathscr{K},\mathscr{L}}}(u_{\mathscr{K}} - u_{\mathscr{L}})(v_{\mathscr{K}} - v_{\mathscr{L}}) + \sum_{\sigma\in\mathscr{E}_{ext}} \frac{m_{\widetilde{\mathbf{e}}}}{d_{\mathscr{K},\mathbf{e}}}(u_{\mathscr{K}} - u_{\mathbf{e}})(v_{\mathscr{K}} - v_{\mathbf{e}})$$

  and $$[\![u_{\partial\mathfrak{M}}, v_{\partial\mathfrak{M}}]\!]_{1,\partial\mathfrak{M}} = \sum_{\mathbf{v}=\mathbf{e}|\mathbf{e}'\in\mathscr{V}} \frac{1}{d_{\mathbf{e},\mathbf{e}'}}(u_{\mathbf{e}} - u_{\mathbf{e}'})(v_{\mathbf{e}} - v_{\mathbf{e}'}).$$

  The associated seminorms are noted $|u_{\mathscr{T}}|_{1,\mathscr{T}}$ and $|u_{\partial\mathfrak{M}}|_{1,\partial\mathfrak{M}}$.

3 Numerical scheme and discrete energy

3.1 Finite-Volume scheme

In this section, we give the finite-volume scheme used to solve the Cahn-Hilliard equation (1). In the interior mesh $\mathfrak{M}$, we use the usual finite-volume approximation based on a consistent two-point flux approximation for Laplace operators. As re-

gards the equation on the boundary mesh $\partial\mathfrak{M}$, we use a $1d$-finite-volume scheme on a curved domain and a consistent two-point flux approximation for the Laplace-Beltrami operator.
We assume that $c^n_{\mathscr{T}} \in \mathbb{R}^{\mathscr{T}}$ is given, the scheme is then written as follows:
Find $(c^{n+1}_{\mathscr{T}}, \mu^{n+1}_{\mathscr{T}}) \in \mathbb{R}^{\mathscr{T}} \times \mathbb{R}^{\mathscr{T}}$ such that $\forall u_{\mathscr{T}}, v_{\mathscr{T}} \in \mathbb{R}^{\mathscr{T}}$:

$$\begin{cases} \left(\dfrac{c^{n+1}_{\mathfrak{M}} - c^n_{\mathfrak{M}}}{\Delta t}, v_{\mathfrak{M}}\right)_{\mathfrak{M}} = -[\![\mu^{n+1}_{\mathscr{T}}, v_{\mathscr{T}}]\!]_{1,\mathscr{T}} \\ \qquad (\mu^{n+1}_{\mathfrak{M}}, u_{\mathfrak{M}})_{\mathfrak{M}} = \displaystyle\sum_{\sigma\in\mathscr{E}_{int}} \frac{m_\sigma}{d_{\mathscr{K},\mathscr{L}}}(c^{n+1}_{\mathscr{K}} - c^{n+1}_{\mathscr{L}})(u_{\mathscr{K}} - u_{\mathscr{L}}) \\ \qquad\qquad + \boxed{\displaystyle\sum_{\sigma\in\mathscr{E}_{ext}} \frac{m_{\widetilde{\mathbf{e}}}}{d_{\mathscr{K},\mathbf{e}}}(c^{n+1}_{\mathscr{K}} - c^{n+1}_{\mathbf{e}})u_{\mathscr{K}}} + \displaystyle\sum_{\mathscr{K}\in\mathfrak{M}} m_{\widetilde{\mathscr{K}}} d^{f_b}(c^n_{\mathscr{K}}, c^{n+1}_{\mathscr{K}})u_{\mathscr{K}} \\ \left(\dfrac{c^{n+1}_{\partial\mathfrak{M}} - c^n_{\partial\mathfrak{M}}}{\Delta t}, u_{\partial\mathfrak{M}}\right)_{\partial\mathfrak{M}} = -[\![c^{n+1}_{\partial\mathfrak{M}}, u_{\partial\mathfrak{M}}]\!]_{1,\partial\mathfrak{M}} - \displaystyle\sum_{\mathbf{e}\in\partial\mathfrak{M}} m_{\widetilde{\mathbf{e}}} d^{f_s}(c^n_{\mathbf{e}}, c^{n+1}_{\mathbf{e}})u_{\mathbf{e}} \\ \qquad\qquad - \boxed{\displaystyle\sum_{\sigma\in\mathscr{E}_{ext}} \frac{m_{\widetilde{\mathbf{e}}}}{d_{\mathscr{K},\mathbf{e}}}(c^{n+1}_{\mathbf{e}} - c^{n+1}_{\mathscr{K}})u_{\mathbf{e}}} \end{cases} \tag{3}$$

With the aim of obtaining convergence result without any condition on the step time Δt, we use a semi-implicit discretization for nonlinear terms:

$$d^{f_b}(x,y) = \frac{f_b(y) - f_b(x)}{y - x} \quad \text{and} \quad d^{f_s}(x,y) = \frac{f_s(y) - f_s(x)}{y - x}, \quad \forall x, y. \tag{4}$$

We can note that we mostly use in practice polynomial functions for f_b and f_s. Then, the term $d^f(x,y)$ can be written as a polynomial function in the variables x, y. Thus, we do not have numerical instability if x is too close to y. If we choose non polynomial functions for nonlinear terms, we have to adapt our discretization (see [1] for more details).
We remark that we can also choose an implicit discretization for nonlinear terms but in that case the same results hold only for $\Delta t \leq \Delta t_0$, with a small enough Δt_0 which only depends on the parameters on the equation.
In each case, we have to use a Newton method at each time step; its convergence is achieved in a few inner iterations.

We can notice that the finite-volume scheme is a low-order method. Thus, the approximation of the boundary does not influence the order of the method and it is not necessary to use curved element to improve the convergence of the scheme (3).

The boxed terms give the coupling between interior and boundary unknowns: the one in the second equation comes from the Laplacian of c in Ω and the one in the third equation stems from the normal derivative term in the dynamic boundary condition on Γ.

In order to improve the presentation and the analysis, we have written the scheme (3) in a way that looks like a variational formulation. We easily recover the usual finite-volume flux balance equations if, for each control volume, we choose the indicator function of this particular control volume as a test function in (3).

3.2 Discrete energy estimate

The discrete energy estimate is one of the key points for the proofs of existence and convergence results.

Definition 1 (Discrete free energy). The discrete free energy corresponding to the continuous definition (2) is defined by:

$$\mathscr{F}_{\mathscr{T}}(c_{\mathscr{T}}) = \underbrace{\frac{1}{2}|c_{\mathscr{T}}|^2_{1,\mathscr{T}} + \sum_{\mathscr{K}\in\mathfrak{M}} m_{\widetilde{\mathscr{K}}} f_b(c_{\mathscr{K}})}_{:=\mathscr{F}_{b,\mathscr{T}}(c_{\mathscr{T}})} + \underbrace{\frac{1}{2}|c_{\partial\mathfrak{M}}|^2_{1,\partial\mathfrak{M}} + \sum_{\mathbf{e}\in\partial\mathfrak{M}} m_{\widetilde{\mathbf{e}}} f_s(c_{\mathbf{e}})}_{:=\mathscr{F}_{s,\partial\mathfrak{M}}(c_{\partial\mathfrak{M}})}, \ \forall c_{\mathscr{T}} \in \mathbb{R}^{\mathscr{T}}.$$

Using the scheme (3) with $u_{\mathscr{T}} = c^{n+1}_{\mathscr{T}} - c^n_{\mathscr{T}}$ and $v_{\mathscr{T}} = \mu^{n+1}_{\mathscr{T}}$ as test functions and the the discretization (4) for nonlinear terms, we obtain the following energy equality:

Proposition 1 (Discrete energy estimate). *Let $c^n_{\mathscr{T}} \in \mathbb{R}^{\mathscr{T}}$. We assume that there exists a solution $(c^{n+1}_{\mathscr{T}}, \mu^{n+1}_{\mathscr{T}})$ to Problem* (3)*. Then, the following equality holds:*

$$\begin{aligned}\mathscr{F}_{\mathscr{T}}(c^{n+1}_{\mathscr{T}}) - \mathscr{F}_{\mathscr{T}}(c^n_{\mathscr{T}}) + \Delta t \left|\mu^{n+1}_{\mathscr{T}}\right|^2_{1,\mathscr{T}} + \frac{1}{\Delta t}\left\|c^{n+1}_{\partial\mathfrak{M}} - c^n_{\partial\mathfrak{M}}\right\|^2_{0,\partial\mathfrak{M}} \\ + \frac{1}{2}\left|c^{n+1}_{\mathscr{T}} - c^n_{\mathscr{T}}\right|^2_{1,\mathscr{T}} + \frac{1}{2}\left|c^{n+1}_{\partial\mathfrak{M}} - c^n_{\partial\mathfrak{M}}\right|^2_{1,\partial\mathfrak{M}} = 0.\end{aligned} \tag{5}$$

This estimate gives a $L^\infty(0,T;H^1(\Omega))$ bound on the discrete solution $c^{\Delta t}_{\mathscr{T}}$ and a $L^\infty(0,T;H^1(\Gamma))$ bound on its trace $c^{\Delta t}_{\partial\mathfrak{M}}$.

4 Convergence analysis

By using the topological degree theory, we can prove that if $c^n_{\mathscr{T}} \in \mathbb{R}^{\mathscr{T}}$ is given, there exists at least one solution $(c^{n+1}_{\mathscr{T}}, \mu^{n+1}_{\mathscr{T}}) \in \mathbb{R}^{\mathscr{T}} \times \mathbb{R}^{\mathscr{T}}$ to discrete Problem (3) (see [8] for more details).

We recall the definition of a solution to Problem (1) in a weak sense:

Definition 2 (Weak formulation).
We say that a couple $(c,\mu) \in L^\infty(0,T;H^1(\Omega)) \times L^2(0,T;H^1(\Omega))$ such that $\mathrm{Tr}(c) \in L^\infty(0,T;H^1(\Gamma))$ is solution to continuous Problem (1) in the weak sense if for all $\psi \in \mathscr{C}^\infty_c\left([0,T[\times\overline{\Omega}\right)$, the following identities hold:

$$\int_0^T \int_\Omega (-\partial_t \psi c + \nabla\mu\cdot\nabla\psi) = \int_\Omega c^0\psi(0,.), \tag{6}$$

$$\begin{aligned}\int_0^T \int_\Omega \left(-\mu\psi + \nabla c\cdot\nabla\psi + f_b'(c)\psi\right) + \int_0^T\int_\Gamma \left(-\partial_t\psi c + \nabla_\parallel c\cdot\nabla_\parallel\psi + f_s'(c)\psi\right) \\ = \int_\Gamma \mathrm{Tr}(c^0)\psi(0,.).\end{aligned} \tag{7}$$

Then, we have the following convergence result.

Theorem 1 (Convergence theorem). *Assuming that Assumptions 1 hold, let us consider Problem* (1) *with an initial condition $c_0 \in H^1(\Omega)$ such that* $\mathrm{Tr}(c_0) \in H^1(\Gamma)$. *Then, there exists a weak solution (c,μ) on $[0,T[$ (in the sense of Definition 2). Furthermore, let $(c^{(m)}, c_{|\Gamma}^{(m)})_{m\in\mathbb{N}}$ and $(\mu^{(m)})_{m\in\mathbb{N}}$ be a sequence of solutions to Problem* (3) *associated with a sequence of discretizations such that the space and time steps, $h_{\mathscr{T}}^{(m)}$ and $\Delta t^{(m)}$ respectively, tend to* 0. *Then, up to a subsequence, the following convergence properties hold, for all $q \geq 1$:*

$$c^{(m)} \to c \text{ in } L^2(0,T;L^q(\Omega)) \text{ strongly}, \quad c_{|\Gamma}^{(m)} \to \mathrm{Tr}(c) \text{ in } L^2(0,T;L^q(\Gamma)) \text{ strongly},$$
$$\text{and} \quad \mu^{(m)} \rightharpoonup \mu \text{ in } L^2(0,T;L^q(\Omega)) \text{ weakly}.$$

The discrete initial concentration used is the mean-value projection.
The main difficulty of this proof is the passage to the limit in nonlinear terms both in Ω and on Γ. Indeed, the usual $L^2((0,T)\times\Omega)$ compactness is not sufficient and we need to have an additional compactness property of the trace of c in $L^2(]0,T[\times\Gamma)$.

Theorem 2 (Estimation of space translates). *There exists an extension operator $\phi : \mathbb{R}^{\mathscr{T}} \to L^2(\mathbb{R}^2)$ satisfying $\phi(u_{\mathscr{T}}) = u_{\mathscr{T}}$ in Ω such that the following identity holds for all $\eta \in \mathbb{R}^2$ with $C > 0$ independent of $h_{\mathscr{T}}$ and η: For all $u_{\mathscr{T}} \in \mathbb{R}^{\mathscr{T}}$,*

$$\|\phi(u_{\mathscr{T}})(.+\eta) - \phi(u_{\mathscr{T}})\|_{L^2(\mathbb{R}^2)}^2 \leq C|\eta|\,(|\eta| + h_{\mathscr{T}})\Big(|u_{\mathscr{T}}|_{1,\mathscr{T}}^2 + |u_{\partial\mathfrak{M}}|_{1,\partial\mathfrak{M}}^2 + \|u_{\partial\mathfrak{M}}\|_{0,\partial\mathfrak{M}}^2\Big).$$

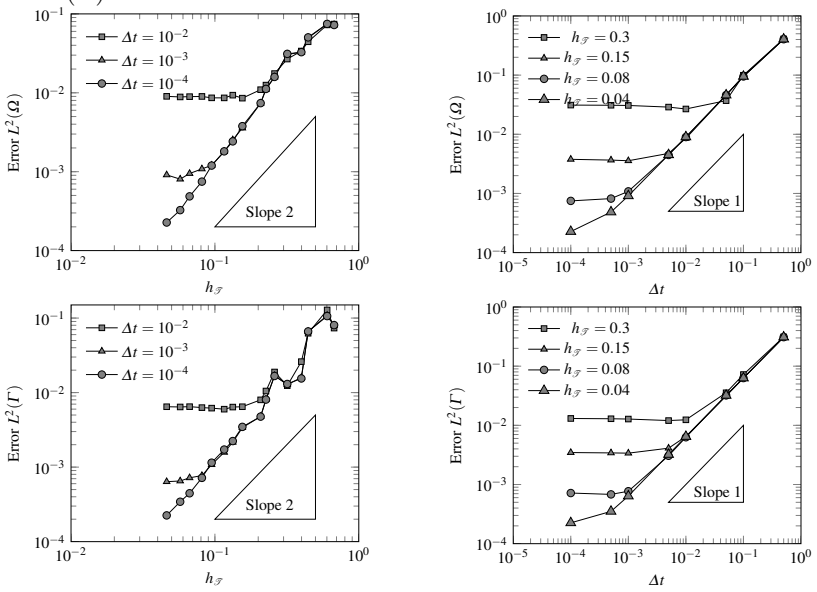
Corollary 1. *Let $(u_{\mathscr{T}_i})_i$ be a sequence of functions with uniform bounds on discrete H^1-norms on Ω and Γ. We can extract a subsequence, still referred to as $(u_{\mathscr{T}_i})_i$ for simplicity, which is strongly converging in $L^2(\Omega)$ towards a certain function u of $H^1(\Omega)$ whose trace belongs to $H^1(\Gamma)$ and such that $(u_{\partial\mathfrak{M}_i})_i$ is strongly converging in $L^2(\Gamma)$ towards* $\mathrm{Tr}(u)$.

To obtain similar results with the sequence of functions which also depends on time, we have to consider the estimation of time translates. To this end, we adapt the proof of Theorem A.2 in [5] and we use the particular form of the extension operator ϕ and the coupling between the domain Ω and its boundary Γ.
Then, thanks to the *a priori* estimates on the solutions (see [8]), there exists $c \in L^2(0,T,H^1(\Omega))$ with $\mathrm{Tr}(c) \in L^2(0,T,H^1(\Gamma))$ such that, up to a subsequence, $c_{\mathfrak{M}}^{\Delta t}$ strongly converges to c in $L^2(]0,T[\times\Omega)$ and moreover, also $c_{\partial\mathfrak{M}}^{\Delta t}$ strongly converges to $Tr(c)$ in $L^2(]0,T[\times\Gamma)$. It is now more or less standard to pass to the limit in the scheme and thus to prove the convergence result.

5 Numerical tests

In [8], we give numerical experiments with different choices of parameters and surface potential f_s that show the different expected qualitative behavior of the solutions. In this paper, we focus on the numerical error estimates. Since no explicit non

trivial solutions are known for our problem, we have to change the Problem (1). We add source term in the first equation of (1) and another one in the third equation of (1). We notice that μ then satisfies a non homogeneous Neumann boundary condition that can be easily handled in the FV setting. We consider the manufactured solution $c(t,(x,y)) = (1+\tanh(5*(x+t))$ with Ω the unit circle. We plot the error between the exact and approximate solutions at time $T = 0.5$ for the norm $L^2(\Omega)$ and $L^2(\Gamma)$.



As expected, we observe the first order convergence in time for the L^2 norm. With respect to the space convergence, as for the Laplace problem we observe a superconvergence phenomenon namely the second order convergence.

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